Classical Geometry Configurations

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1 Notation

Throughout this note, ABC will denote a triangle with incenter I, circumcircle Γ, circumcenter O, orthocenter H and excenters I_a, I_b and I_c. We will also denote the angles $\angle A, \angle B$ and $\angle C$ by a, b and c, respectively. We won't worry too much about configuration issues in this note (but you should in your write-ups!). Many of the lemmas and configurations presented here are from Yufei Zhao's note Lemmas in Euclidean Geometry, which is a great reference on the subject.

2 An Example Lemma

There are many lemmas that come up in various Olympiad geometry problems, the vast majority of which are not known by a name or deemed to be "well-known". In this note, we will try to sample a mixture of well-known and relatively unknown lemmas. Some of these lemmas are quite easy to prove and are primarily useful because they give you one more thing that is true for free when solving a problem. Other lemmas would themselves yield quite good Olympiad problems and involve a wide range of techniques.

The first lemma we present has a nice proof making use of one of the most important facts in Olympiad geometry – spiral similarity. This says that if OAB and OCD are similar triangles with the same orientation, then OAC and OBD are also similar.

Example 1. *Let D be the point at which the incircle is tangent to BC and suppose that P and Q lie on segments BI and CI such that $\angle PAQ = \frac{1}{2}\angle BAC$. Then $\angle PDQ = 90°$.*

Proof. Let E be the intersection of the line perpendicular to BI through P with AB. Define F similarly on AC. Note that BEP and BID are similar right triangles, implying by spiral similarity that BEI and BPD are similar. Thus $\angle BDP = \angle BIE$. Similarly $\angle CDQ = \angle CIF$. Thus it suffices to show that $\angle BIE + \angle CIF = 90°$.

Now note that $\angle AEP = 90° + b/2 = \angle AIQ$. Note that $\angle PAQ = \frac{1}{2}\angle BAC$ implies that $\angle EAP = \angle IAQ$. Thus AEP and AIQ are similar, implying by spiral similarity that AEI and APQ are similar. By symmetry, these are both also similar to AIF. Now it follows that $\angle BIC + \angle EIF = \angle BIC + \angle APQ + \angle AQP = 90° + a/2 + 180° - a/2 = 270°$, implying $\angle BIE + \angle CIF = 90°$. □

The main motivation for adding the points E and F is that we first notice $\angle EAP = \angle IAQ$, which means there is a hidden spiral similarity about A mapping these angles to one another. This prompts us to at least try to "complete the transformation". A natural way to do this is to consider the spiral similarity mapping AQ to AP and to define E as the point which I is mapped to. After we have defined E in this way, everything falls into place. Completing transformations is a really useful way (but certainly not the only way) to motivate adding "magic points" that solve diagrams. A lot of key points are either the centers of transformations existing in the diagram or the images of points under transformations. Spiral similarities are always present, rotations are present whenever there is an isosceles triangle, homotheties are present in all trapezoids and tangent circles and translations are present in any parallelogram.

We next will give an alternative proof which is admittedly contrived and far less motivated. However, it at least proves a little bit more and explores a slightly different configuration. It also demonstrates the phantom point method, where we define a point D' alternatively in a way that is easier to use and then show that $D = D'$.

Alternative Proof. Consider the point F on BC such that AP bisects $\angle BAF$. It follows since $\angle PAQ = \frac{1}{2}\angle BAC$ that AQ bisects $\angle CAF$. Thus P and Q are the incenters of BAF and CAF. Let the other internal tangent between the incircles of BAF and CAF intersect BC again at D'. Suppose that the incircles of BAF and CAF are tangent to BC at X and Y. Note that $FX - FY$ is the length of the common tangent between the two incircles and thus equal to $D'Y - D'X$. Since $D'X + D'Y = XY = FX + FY$, it follows that $FY = D'X$. Therefore $BD' = BX + FY = \frac{1}{2}(AB + BF - AF) + \frac{1}{2}(AF + FC - AC) = \frac{1}{2}(AB + BC - AC) = BD$. Thus $D = D'$. Since PD and QD bisect the angles formed by the other common tangent to the incircles of BAF and CAF and BC, it follows that $\angle PDQ = 90°$. □

3 Incenter, Excenters and Midpoints of Arcs

In this section, we go over a few triangle lemmas and results that come up fairly frequently. We begin with a classical lemma.

Lemma 1. *Suppose that D is the midpoint of the arc $\widehat{BC}$ not containing A of Γ, then D is the midpoint of II_a and the center of a circle passing through B, I, C and I_a.*

Proof. Since A, I and D are collinear, a quick angle chase yields that $\angle DIB = \angle DBI = 90° - c/2$. Similarly, an angle chase yields $\angle DI_aB = \angle DBI_a = c/2$, implying the result. □

The next example we will show illustrates the usefulness of power of a point. In this example, it initially seems hard to relate I to either M or E. Power of a point can in general be useful for finding angles about seemingly unrelated points.

Example 2. *Suppose that E is the midpoint of arc $\widehat{BAC}$ and M is the midpoint of side BC. Then AI is tangent to the circumcircle of EIM.*

Proof. Let D be the midpoint of arc $\widehat{BC}$. Note that DEB is a right triangle since DE is a diameter of Γ and M is the projection of B onto DE, which yields that $DB^2 = DM \cdot DE$. It follows by the previous lemma now that $DB = DI$ and hence $DI^2 = DB^2 = DM \cdot DE$, implying the result by power of a point. □

Next we prove a classical lemma by completing a homothety present in the diagram – another "completing the transformation" style proof. In general, whenever there are circles tangent to sides, tangent circles, it is useful to consider completing homotheties.

Lemma 2. *Suppose that incircle and A-excircle of ABC are tangent to BC at M and N, then AN passes through the point diametrically opposite to M on ω and AM passes through the point diametrically opposite to N on ω_a.*

Proof. Consider the homothety with center A mapping the A-excircle to the incircle of ABC. This maps the point N to N' on the incircle such that the tangent to the incircle at N' is parallel to BC. This is the point diametrically opposite M. The other result follows similarly. □

In applications of power of a point or homothety, it often can be useful to introduce new circles as in the following example.

Example 3. *Let D be the foot of the altitude from A to BC, let M be the midpoint of AD and let K be the point of tangency between the incircle of ABC and BC. Then I_a, K and M are collinear.*

Proof. Consider the circle ω with diameter AD. Let P be the point diametrically opposite to the point of tangency between the A-excircle and BC. Note that AP must intersect BC at the center of the internal homothety mapping ω to the A-excircle. By the previous lemma, AP intersects BC at K. The result follows since the centers of the two circles and K must be collinear. □

Sometimes a line, point or other object is in a kind of awkward place and it would be much more convenient to move it elsewhere i.e. the angles around some point are completely unrelated to the rest of the diagram. Applying some sort of transformation (a homothety, rotation, spiral similarity or translation) is usually the right way to move it around. The key is to move the object around in such a way that its important properties are preserved. In the next lemma, a segment is in the wrong place, so we move it to a more convenient place with a homothety. The key is we only care about ratios of its subsegments, which are preserved under homotheties.

Lemma 3. *Suppose that the incircle of ABC is tangent to BC, AC and AB at D, E and F. Let M be the midpoint of BC. The perpendicular to BC at D, the median AM and the line EF are concurrent.*

Proof. Consider the line parallel to BC passing through the intersection P of AM and EF. Let it intersect AB and AC at Q and R. It follows that P is the midpoint of QR since QR is parallel to BC. Let the line parallel to AC through Q intersect EF at S. It follows that $QSRF$ is a parallelogram since SF bisects QR and SQ is parallel to AC. Since EAF is isosceles, so is SQE. Therefore $QE = QS = RF$. Now suppose that QR intersects the incircle of ABC at U and V. By power of a point, we now have that $QU \cdot QV = QE^2 = RF^2 = RV \cdot RU$. Since $QV - QU = RV = RU$, we have that $QV = RU$ and thus P is also the midpoint of UV. This implies that IP is perpendicular to UV and therefore BC, implying the result. □

Our last lemma of this section, is Euler's formula. Note that a fun corollary of Euler's formula is that $R \geq 2r$. This is another great example of power of a point.

Lemma 4. *(Euler's Formula) Let ABC have circumradius R, inradius r and A-exradius r_a. Then*

1. $OI = \sqrt{R(R-2r)}$.
2. $OI_a = \sqrt{R(R+2r_a)}$.

Proof. We just prove 1, as the proof of 2 is similar. Let D and E be the midpoints of arcs $\widehat{BC}$ and $\widehat{BAC}$ of Γ. Let F be the point at which the incircle of ABC is tangent to AC. Both triangles EDC and AIC are right triangles with angle $\angle IAF = \angle EDC = a/2$. Therefore $AI/IF = DE/DC$. By the first lemma, $DC = DI$. Substituting $DE = 2R$ and $IF = r$ yields that $AI \cdot ID = 2Rr$. Now power of a point applied to I with respect to Γ implies that $R^2 - OI^2 = AI \cdot ID = 2Rr$. $\square$

More Lemmas. The proofs are left to you as exercises.

1. The intersections of the internal and external bisectors of $\angle BAC$ with the perpendicular bisector of BC lie on Γ. This is a common trick to show four points are concyclic.

2. If the incircle and A-excircle of ABC are tangent to BC at D and E, $BD = CE$.

3. If M is the midpoint of arc $\widehat{BAC}$ of Γ, then M is the midpoint of I_bI_c and the center of the circle through I_b, I_c, B and C.

4. Let D be the midpoint of the arc $\widehat{BC}$ not containing A of Γ. Suppose that two lines through D intersect BC at P_1 and P_2 and Γ at Q_1 and Q_2. Show that $P_1P_2Q_1Q_2$ is cyclic and DI is tangent to the circumcircle of P_1IP_2.

5. Suppose that E is the midpoint of the arc $\widehat{BAC}$ of Γ. Suppose P and Q are the points at which the B-excircle and C-excircle touch AC and AB, respectively. Show that $EP = EQ$.

6. Let D and E be the midpoints of arcs $\widehat{AB}$ and $\widehat{AC}$ of Γ, and let P be the midpoint of arc $\widehat{BAC}$. Show that $DAEI$ is a kite and $DPEI$ is a parallelogram.

7. Suppose that the incircle ω of ABC is tangent to BC, AC and AB at D, E and F. Then angle bisector CI intersects FE at a point T on the line adjoining the midpoints of AB and BC. It also holds that $BFTID$ is cyclic and $\angle BTC = 90°$.

8. Suppose that D, E and F are the points at which the incircle of ABC touches AB, AC and BC. Let P be the point at which AF intersects the incircle again. Show that the tangent to the incircle at P, EF and BC are concurrent.

9. Suppose that D, E and F are the points at which the incircle of ABC touches AB, AC and BC. Let P be the point at which DE intersects BC. Show that the radical axis between the incircle and the circle with diameter II_a bisects PF.

4 More Triangle Lemmas

Often in problems involving triange centers, points such as the midpoints of the sides, midpoints of arcs, feet of the altitudes, I, O, H, and intersections of AH, BH and CH with Γ are implicitly present. It is worthwhile always checking if drawing in these standard points are useful. It is also important to always look for similar triangles, angle chase and remember to use power of a point when applicable – i.e. always length and angle chase completely.

We now will prove some standard results about the symmedian. Here we construct two points E and F which are essentially B and C inverted about A with a certain radius. Applying inversions or homotheties coupled with reflections to produce antiparallel lines is a tool that comes up every now and then.

Lemma 5. *(Symmedian) If M is the midpoint of BC, then the symmedian from A is defined to be the line that is the reflection of AM in the bisector of angle $\angle BAC$.*

1. *If the tangents to Γ at B and C intersect at N, then N lies on the symmedian from A and therefore $\angle BAM = \angle CAN$.*
2. *If the symmedian from A intersects Γ at D, then $AB/BD = AC/CD$.*
3. *Let ω_b be the circle passing through A, B and tangent to AC at A. Define ω_c similarly. Then ω_b and ω_c intersect again on the A-symmedian.*

Proof. Consider the line antiparallel to BC passing through N. Let this line intersect AB and AC at E and F. In other words, let E and F be such that $\angle AEF = c$ and $\angle AFE = b$ and N lies on EF. Angle chasing yields that $\angle EBN = c$ and $\angle FCN = b$. Therefore ENB and FNC are isosceles, implying $NE = NB = NC = NF$. Thus N is the midpoint of FE. It follows that $ABMC$ is similar to $AFNE$ and thus AM and AN are reflections about the bisector of $\angle BAC$, proving 1. Now some algebra with ratios from similar triangles gives that $AB^2/BD^2 = ND/NA = AC^2/CD^2$, which proves 2.

Let NB intersect ω_b again at P and NC intersect ω_c again at Q. Some angle chasing yields that PAB is similar to ACB and QCA is similar to ABC. It then follows that P, A and Q are collinear and that $PQBC$ is cyclic. Therefore N has the same power of a point with respect to ω_b and ω_c and thus lies on the radical axis of the two circles, proving 3. $\square$

Next we prove a classical lemma about the orthocenter.

Lemma 6. *If D is the point diametrically opposite to A on Γ and M is the midpoint of BC, then M is also the midpoint of HD. If E and F are the intersections of AH with BC and Γ, respectively, then E is the midpoint of HF.*

Proof. Angle chasing yields that $\angle HBE = \angle HBF = 90^\circ - c$ and thus it follows that E is the midpoint of HF since BE is perpendicular to HF. Angle chasing also yields that $BHCD$ is a parallelogram, implying that M is the midpoint of HD. $\square$

We won't prove the last three lemmas of this section, but we highlight them apart from the additional lemmas we list afterwards. I encourage you to prove Lemma 9 on your own.

Lemma 7. *(Steiner Line) Suppose that D lies on Γ and P, Q and R are the reflections of D in sides AB, AC and BC. Then P, Q and R are collinear and H lies on line PQR.*

Lemma 8. *(Center of Spiral Similarity) Let AB and CD be two segments, and let lines AC and BD meet at X. Let the circumcircles of ABX and CDX meet again at O. Then O is the center of the spiral similarity that carries AB to CD.*

Whenever trying to prove a set of circles all have a common point, that point is often the center of some family of spiral similarities.

Lemma 9. *(Example Families of Spiral Similarities)*

1. *Suppose two rays ℓ_1 and ℓ_2 both have endpoint A and let P be a fixed point. Let ω be a circle passing through A and P, and let this circle intersect ℓ_1 and ℓ_2 at E and F. As ω varies over all such circles, the segments EF are spirally similar with center P.*

2. *Suppose that P and Q are on sides AB and AC and are such that $BP/PA = AQ/QC$. All such segments PQ are spirally similar with center on the A-symmedian of ABC.*

3. *Let c be a constant. Suppose two rays ℓ_1 and ℓ_2 both have endpoint A. The segments BC with B on ℓ_1, C on ℓ_2 such that $AB + AC = c$ are all spirally similar with a common center of spiral similarity that lies on the bisector of the angle formed by ℓ_1 and ℓ_2.*

More Lemmas. The proofs are left to you as exercises.

1. If BH and CH intersect AC and AB at D and E, and M is the midpoint of BC, then M is the center of the circle through B, D, E and C, and MD and ME are tangent to the circumcircle of ADE.

2. If M is the midpoint of BC then $AH = 2 \cdot OM$.

3. (Euler Line) If O, H and G are the circumcenter, orthocenter and centroid of a triangle ABC, then G lies on segment OH with $HG = 2 \cdot OG$.

4. If AH, BH and CH intersect Γ again at D, E and F, then there is a homothety centered at H sending the triangle formed by projecting H onto the sides of ABC to DEF with ratio 2.

5. If D, E and F are on Γ then AD, BE and CF are concurrent if and only if

6. (Nine-Point Circle) Let ω denote the circle passing through the midpoints of the sides of ABC. Then ω passes through the midpoints of AH, BH and CH and the projections of H onto the sides of ABC.

7. Γ is the nine-point circle of the triangle $I_aI_bI_c$.

8. Let D and E be the foot of altitudes from B and C to AC and AB. If the circumcircle of ADE intersects Γ again at P, then P, H and M are collinear. Furthermore, AP, DE and BC are concurrent.

9. Suppose that BI and CI intersect AC and AB at P and Q. Then PQ is the radical axis between Γ and the circumcircle of I_bII_c.

10. Let D, E and F be the feet of the altitudes from A, B and C in triangle ABC. Let M be the midpoint of BC and EF intersect Γ again at X and Y. Then X, Y, D and M are concyclic and if XM and YM intersect the nine-point circle again at P and Q, then the center of spiral similarity mapping PQ to XY is D.

11. Let P be inside ABC satisfying that $AP \cdot BC = BP \cdot AC = CP \cdot AB$. Show that $\angle BPC = \angle A + 60^\circ$, $\angle APB = \angle C + 60^\circ$ and $\angle APC = \angle B + 60^\circ$.

12. (Fermat Point) Let P, Q and R be outside ABC and satisfying that PAB, QAC and RBC are equilateral. Show that AR, BQ and CP are concurrent.

5 Circles Tangent to Γ

In this section, we cover a few results about circles tangent to Γ, which have recently become popular because they are difficult to bash. Often problems about tangent circles have solutions using Casey's theorem, but we will focus on more elegant methods here.

Whenever there are two tangent circles, there is a homothety mapping one to the other centered at the point of tangency. Completing these homotheties can be useful, especially when the circle is also tangent to lines in the diagram. Here is a classical result on this topic.

Lemma 10. *Suppose that ω is tangent to Γ at A and to BC at D. Then AD bisects angle $\angle BAC$.*

Proof. Let AD intersect Γ at P. Consider the homothety with center A mapping ω to Γ. This maps the line BC to the line tangent to Γ at D. Since these two lines are parallel, P must be the midpoint of arc $\widehat{BC}$. Thus the line ADP bisects $\angle BAC$. □

A corollary of this result and Pascal's theorem is as follows.

Lemma 11. *(Mixtilinear Incircles) Let ω be a circle tangent internally to Γ and to AB and AC at X and Y. Then I is the midpoint of segment XY.*

Proof. Let ω be tangent to AB and AC at P and suppose PX and PY intersect Γ again at X' and Y'. By Lemma 9, X' and Y' are the midpoints of arcs $\widehat{AB}$ and $\widehat{AC}$, implying that I is the intersection of CX' and BY'. It follows that X, I and Y are collinear by Pascal's theorem applied to $AY'CPBX'$. Since AI bisects $\angle BAC$ and XAY is isosceles, I must be the midpoint of XY. □

This next example is a difficult lemma illustrating several points. This proof we present is based on the proof in Yufei Zhao's handout on Lemmas in Euclidean Geometry.

- Phantom points: Working backwards, we see that if the result is true then $AFIM$ must be cyclic. But it's hard to do anything with this directly so we define F differently so that it's easier to work with. Often its useful to define points as intersections with circles to get nice angle relationships.
- Power of a point is a great way to get angle relationships between points that otherwise are difficult to relate. Here we get nice angles around I to E and M using power of a point.
- Points of tangency between circles are often on the circumcircle of other points in the diagram. In this case, M is on the circumcircle of AIF.

Lemma 12. *(Curvilinear Incircles) Let D be an arbitrary point on segment BC. Let ω be a circle tangent to Γ, DA and DC. If ω is tangent to DA and DC at F and E, then I lies on FE.*

Proof. Let M be the point of tangency between ω and Γ and let K be the midpoint of arc $\widehat{BC}$. We have that A, I and K are collinear and M, E and K are collinear by the previous lemma. Now let F' be the second intersection of EI with the circumcircle of AMI. Observe that $\angle MF'E = \angle MAI = \frac{1}{2}\widehat{MK} = \frac{1}{2}\widehat{ME}$ by the fact that $AMIF'$ is cyclic and ω and Γ are homothetic with center M. This implies that F' is on ω. Angle chasing gives that KEC and KCM are similar and therefore $KI^2 = KC^2 = KE \cdot KM$, which implies that KIM and KEI are similar. Therefore $\angle AF'M = \angle AIM = \angle IEK = \angle F'EK$, implying that AF' is tangent to ω. Thus $F' = F$, proving the result. □

In general when working with tangent circles, it can also be useful to draw the common tangent line at the point of tangency and consider its intersection P with some other line. This may allow you to define the tangency point using power of a point relations or as the intersection of a circle centered at P with Γ.

A notable problem where the key was to realize that the point of tangency lies on other circumcircles is IMO 2011 #6. We sketch a solution to that problem here. In general it is fairly hard to figure out what is the right circle to choose. I suggest looking for cyclic quadrilaterals in the diagram and trying to guess them based on what would be convenient and yield useful angles.

Example 4. *(IMO 2011) Let ABC be an acute triangle with circumcircle Γ. Let ℓ be a tangent line to Γ, and let ℓ_a, ℓ_b and ℓ_c be the lines obtained by reflecting ℓ in the lines BC, CA and AB, respectively. Show that the circumcircle of the triangle determined by the lines ℓ_a, ℓ_b and ℓ_c is tangent to the circle Γ.*

Proof Sketch. Let A', B' and C' be the intersections of ℓ_b and ℓ_c, ℓ_a and ℓ_c, and ℓ_a and ℓ_b, respectively. Let P be the point of tangency between Γ and ℓ and let Q be the reflection of P through BC. Now let T be the second intersection of the circumcircles of $BB'Q$ and $CC'Q$. It can be shown that T lies on Γ and the circumcircle of $A'B'C'$ by angle chasing. Similarly, T can be shown to be a point of tangency between the circles by angle chasing. The angle chasing is made easier by first showing that AA', BB' and CC' meet at the incenter I of $A'B'C'$. $\square$

More Lemmas. The proofs are left to you as exercises.

1. Let D be the midpoint of arc $\widehat{BAC}$ and let M be the point at which the circle tangent to AB, AC and Γ is tangent to Γ. Show that D, I and M are collinear.

2. Using the same notation as in 1, let E be the point at which the incircle of ABC is tangent to BC. If ME intersects Γ again at F, show that AF is parallel to BC.

6 Some Takeaways

- Figure out what's true: many geometry problems will involve proving an intermediate result.
  1. Draw at least one precise diagram, draw in relevant circles and extend lines. Look for concurrencies.
  2. Look for quadrilaterals that might be cyclic.
  3. Work backwards. What would imply the result? What would be convenient if true?
- Do everything straightforward:
  1. Angle chase completely, look for similar triangles and apply power of a point.
  2. Draw in implicit points: the midpoints of the sides, midpoints of arcs, feet of the altitudes, I, O, H, the intersections of AH, BH and CH with Γ, etc.
- Relate the unrelated with power of a point.
- Complete transformations: spiral similarities, homotheties, translations and rotations.

1. Draw in the images of points under these transformations.
2. Draw in the center of the transformation.
3. Move angles or segments to more convenient places.

- When there are midpoints: consider homotheties with ratio 2, add more midpoints, complete parallelograms.
- Intersect lines and circumcircles to get angle relationships about points.
- Tangency points between circles:
  1. Consider homotheties about the tangency point.
  2. Draw the common tangent line, intersect it with some other line P and define the tangency point using power of a point.
  3. Look for a circle or some triangle PQR such that the circumcircle of PQR passes through the tangency point.
- Phantom points: figure out something true and redefine a point P in an easier way as P'. Prove that $P = P'$. Often it is useful to define P' as the intersection of a line with a circumcircle to get angle relationships about P'.
- Mysterious perpendicular lines sometimes can be dealt with by introducing circles centered on one line in order to make the other their radical axis.

7 Other Classical Configurations

Here is a selection of a few other lemmas and configurations that come up often in Olympiads.

1. Let $ABCD$ be a cyclic quadrilateral such that AB and CD intersect at P and diagonals AC and BD intersect at Q. Then:
$$\frac{BQ}{QD} = \frac{AB \cdot BC}{AD \cdot DC} \quad \text{and} \quad \frac{PB}{PA} = \frac{BC \cdot BD}{AC \cdot AD}$$
2. If $ABCD$ is a quadrilateral such that $\angle BCD = 90^\circ + \frac{1}{2}\angle DAB$ then it follows that A is the circumcenter of BCD.
3. (Pole-Polar) Let X lie on the line joining the points of tangency of the tangents from Y to a circle Ω. Then Y lies on the line joining the points of tangency of the tangents from X to Ω.
4. (Ceva's Theorem) Let ABC be a triangle and D, E and F be on the lines BC, AC and AB such that an even number are on the extensions of the sides (zero or two). Then AD, BE and CF are concurrent if and only if
$$\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = 1$$

5. (Menelaus' Theorem) Let ABC be a triangle and D, E and F be on the lines BC, AC and AB such that an odd number are on the extensions of the sides (one or three). Then D, E and F are collinear if and only if
$$\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = 1$$

6. (Trig Ceva) Let ABC be a triangle and D, E and F be on the lines BC, AC and AB such that an even number are on the extensions of the sides (zero or two). Then AD, BE and CF are concurrent if and only if
$$\frac{\sin(\angle ABE)}{\sin(\angle CBE)} \cdot \frac{\sin(\angle BCF)}{\sin(\angle ACF)} \cdot \frac{\sin(\angle CAD)}{\sin(\angle BAD)} = 1$$

7. (Casey's Theorem) If A_1, B_1 and C_1 are points on the sides BC, AC and AB of a triangle ABC, then the perpendiculars to their respective sides at these three points are concurrent if and only if $BA_1^2 - CA_1^2 + CB_1^2 - AB_1^2 + AC_1^2 - BC_1^2 = 0$.

8. (Apollonius Circle) Let ABC be a given triangle and let P be a point such that $AB/BC = AP/PC$. If the internal and external bisectors of angle $\angle ABC$ meet line AC at Q and R, then P lies on the circle with diameter QR.

9. Let $ABCD$ be a convex quadrilateral. The four interior angle bisectors of $ABCD$ are concurrent and there exists a circle Γ tangent to the four sides of $ABCD$ if and only if $AB + CD = AD + BC$.

10. (Simson Line) Let M, N and P be the projections of a point Q onto the sides of a triangle ABC. Then Q lies on the circumcircle of ABC if and only if M, N and P are collinear. If Q lies on the circumcircle of ABC, then the reflections of Q in the sides of ABC are collinear and pass through the orthocenter of the triangle.

11. (Radical Axis to a Point) Suppose that Γ is a circle and P and Q are points such that P lies on line passing through the midpoints of the tangents from Q to Γ, then the length of the tangent from P to γ is equal to PQ.

12. (Monge's Theorem) Given three circles Γ_1, Γ_2 and Γ_3. If P, Q and R are the external centers of homothety between pairs of the three circles, then P, Q and R are collinear. If P and Q are internal centers of homothety, then P, Q and R are also collinear.

13. (Pascal's Theorem) If A, B, C, D, E, F are points on a circle then the intersections of the pairs of lines AB and DE, BC and EF, CD and FA lie on a line.

14. Pascal's theorem is true when points are not necessarily distinct and many of its applications concern tangent lines when some of the six points are equal.

15. (Pappus' Theorem) If A, C and E lie on one line ℓ_1 and B, D and F lie on a line ℓ_2, then the intersections of the pairs of lines AB and DE, BC and EF, CD and FA lie on a line.

16. (Brianchon's Theorem) If $ABCDEF$ is a hexagon with an inscribed circle then AD, BE and CF are concurrent.

17. (Desargues Theorem) Let ABC and XYZ be triangles. Let D, E, F be the intersections of the pairs of lines AB and XY, BC and YZ, AC and XZ. Then D, E and F are collinear if and only if AX, BY and CZ are concurrent.

18. (Casey's Theorem) Let O_1, O_2, O_3, O_4 be four circles tangent to a circle O. Let t_{ij} be the length of the external common tangent between O_iO_j if O_i and O_j are tangent to O from the same side and the length of the internal common tangent otherwise. Then

$$t_{12} \cdot t_{34} + t_{41} \cdot t_{23} = t_{13} \cdot t_{24}$$

    The converse is also true: if the above equality holds then O_1, O_2, O_3, O_4 are tangent to O.

19. (Simson Line) Let M, N and P be the projections of a point Q onto the sides of a triangle ABC. Then Q lies on the circumcircle of ABC if and only if M, N and P are collinear. If Q lies on the circumcircle of ABC, then the reflections of Q in the sides of ABC are collinear and pass through the orthocenter of the triangle.

20. (Butterfly Theorem) Let M be the midpoint of a chord XY of a circle Γ. The chords AB and CD pass through M. If AD and BC intersect chord XY at P and Q, then M is also the midpoint of PQ.

21. (Broken Chord Theorem) Let E is the midpoint of major arc $\widehat{ABC}$ of the circumcircle of a triangle ABC where $AB < BC$. If D is the projection of E onto BC, then $AB + BD = DC$.

22. (Feuerbach's Theorem) The nine-point circle is tangent to the incircle and excircles.

23. (Miquel Point) Let D, E and F be points on sides BC, AC and AB of a triangle ABC. Then the circumcircles of AEF, BDF and CDE are concurrent.

24. (Isogonal Conjugates) Let ABC be a triangle and P be a point. If the reflection of BP in the angle bisector of $\angle ABC$ and the reflection of CP in the angle bisector $\angle ACB$ intersect at Q, then Q lies on the reflection of CP in the angle bisector of $\angle ACB$.

8 Problems

The problems here are a few examples from past IMO Shortlists. Some directly use the lemmas above while others do not.

1. Given three fixed pairwisely distinct points A, B, C lying on one straight line in this order. Let G be a circle passing through A and C whose center does not lie on the line AC. The tangents to G at A and C intersect each other at a point P. The segment PB meets the circle G at Q. Show that the point of intersection of the angle bisector of the angle AQC with the line AC does not depend on the choice of the circle G.

2. Let $ABCD$ be a cyclic quadrilateral whose diagonals AC and BD meet at E. The extensions of the sides AD and BC beyond A and B meet at F. Let G be the point such that $ECGD$ is a parallelogram, and let H be the image of E under reflection in AD. Prove that D, H, F, G are concyclic.

3. A convex quadrilateral $ABCD$ has perpendicular diagonals. The perpendicular bisectors of the sides AB and CD meet at a unique point P inside $ABCD$. Prove that the quadrilateral $ABCD$ is cyclic if and only if triangles ABP and CDP have equal areas.

4. Let $ABCD$ be a fixed convex quadrilateral with $BC = DA$ and BC not parallel with DA. Let two variable points E and F lie of the sides BC and DA, respectively and satisfy $BE = DF$. The lines AC and BD meet at P, the lines BD and EF meet at Q, the lines EF and AC meet at R. Prove that the circumcircles of the triangles PQR, as E and F vary, have a common point other than P.

5. Let O be the circumcenter and H the orthocenter of an acute triangle ABC. Show that there exist points D, E, and F on sides BC, CA, and AB respectively such that
$$OD + DH = OE + EH = OF + FH$$
and the lines AD, BE, and CF are concurrent.

6. In an acute triangle ABC the points D, E and F are the feet of the altitudes through A, B and C respectively. The incenters of the triangles AEF and BDF are I_1 and I_2 respectively; the circumcenters of the triangles ACI_1 and BCI_2 are O_1 and O_2 respectively. Prove that I_1I_2 and O_1O_2 are parallel.

7. Let ABC be a triangle with $AB \neq AC$ and circumcenter O. The bisector of $\angle BAC$ intersects BC at D. Let E be the reflection of D with respect to the midpoint of BC. The lines through D and E perpendicular to BC intersect the lines AO and AD at X and Y respectively. Prove that the quadrilateral $BXCY$ is cyclic.

8. Let ABC be a triangle, and M the midpoint of its side BC. Let γ be the incircle of triangle ABC. The median AM of triangle ABC intersects the incircle γ at two points K and L. Let the lines passing through K and L, parallel to BC, intersect the incircle γ again in two points X and Y. Let the lines AX and AY intersect BC again at the points P and Q. Prove that $BP = CQ$.

9. Let ABC be an acute triangle with circumcircle Ω. Let B_0 be the midpoint of AC and let C_0 be the midpoint of AB. Let D be the foot of the altitude from A and let G be the centroid of the triangle ABC. Let ω be a circle through B_0 and C_0 that is tangent to the circle Ω at a point $X \neq A$. Prove that the points D, G and X are collinear.

10. Let $ABCDE$ be a convex pentagon such that $BC \parallel AE$, $AB = BC + AE$, and $\angle ABC = \angle CDE$. Let M be the midpoint of CE, and let O be the circumcenter of triangle BCD. Given that $\angle DMO = 90^\circ$, prove that $2\angle BDA = \angle CDE$.

11. Let AH_1, BH_2, CH_3 be the altitudes of an acute angled triangle ABC. Its incircle touches the sides BC, AC and AB at T_1, T_2 and T_3 respectively. Consider the symmetric images of the lines H_1H_2, H_2H_3 and H_3H_1 with respect to the lines T_1T_2, T_2T_3 and T_3T_1. Prove that these images form a triangle whose vertices lie on the incircle of ABC.

12. Let $ABCDEF$ be a convex hexagon all of whose sides are tangent to a circle ω with centre O. Suppose that the circumcircle of triangle ACE is concentric with ω. Let J be the foot of

the perpendicular from B to CD. Suppose that the perpendicular from B to DF intersects the line EO at a point K. Let L be the foot of the perpendicular from K to DE. Prove that $DJ = DL$.

13. Let $ABCD$ be a circumscribed quadrilateral. Let g be a line through A which meets the segment BC in M and the line CD in N. Denote by I_1, I_2 and I_3 the incenters of $\triangle ABM$, $\triangle MNC$ and $\triangle NDA$, respectively. Prove that the orthocenter of $\triangle I_1I_2I_3$ lies on g.

9 Hints

1. (2003 G2) Introduce the other intersection of PB with the circle. Use similar triangles to find useful ratios of sides and do a bit of algebra.

2. (2012 G2) What pair of similar triangles would imply that D, H, F and G are concyclic?

3. (1998 G1) Let Q be the intersection of the diagonals and think about $MPQN$ where M and N are the midpoints of AD and BC.

4. (2005 G4) What transformations are present in the diagram? Define the center of this transformation.

5. (2000 G3) Remember the lemma that the reflection of H in the line BC lies on Γ.

6. (2012 G3) Draw in the circumcircles AI_1C and BI_2C. What do you notice? Now assume the desired result and work backwards to figure out what is true.

7. (2012 G4) Draw in the midpoints of arcs $\widehat{BC}$ and $\widehat{BAC}$. This is in general a good idea whenever there is an incenter, angle bisector or sometimes even the circumcenter or midpoint of a side.

8. (2005 G6) Try to reduce the problem to a result not involving P, Q, X or Y. Is any of the lemmas from this handout particularly useful?

9. (2011 G4) Where does the tangent to Ω at X intersect B_0C_0? Are there any more natural points to introduce into the diagram?

10. (2010 G5) Two general principles for creating new points to make use of midpoints are: (1) reflect points through a midpoint to produce a parallelogram, and (2) add in more midpoints. Whenever you are given a sum of lengths condition such as $AB = BC + AE$, it is often useful to try construct the sum of length e.g. create a segment of length $BC + AE$ by adding a new point to the diagram. Try applying all of this here.

11. (2000 G8) Figure out the orientations of the sides of the triangle and reverse engineer H_i from the points of the triangle. Show that it suffices to prove that these phantom points H_i' are on the tangents to the incircle at T_i. Rephrasing what you want to prove, you should arrive at a statement involving a triangle PQR, the midpoints of the major arcs of its circumcircle and reflections of lines intersecting on a tangent to the circumcircle.

12. (2011 G7) The difficult line here is the perpendicular from B to DF. Try to make this into the radical axis between two circles. In general it is worth trying to make mysterious perpendicular lines into radical axes by introducing circles.

13. (2009 G8) What point in the diagram might lie on the circumcircle of $I_1I_2I_3$? Prove it.